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On T-duality and integrability for strings on AdS backgrounds

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Abstract: We discuss an interplay between T-duality and integrability for certain classical non-linear sigma models. In particular, we consider strings on the $\mathrm{AdS}_5 \times S^5$ background and perform T-duality along the four isometry directions of AdS_5 in the Poincaré patch. The T-dual of the AdS_5 sigma model is again a sigma model on an AdS_5 space. This classical T-duality relation was used in the recently uncovered connection between light-like Wilson loops and MHV gluon scattering amplitudes in the strong coupling limit of the AdS/CFT duality. We show that the explicit coordinate dependence along the T-duality directions of the associated Lax connection (flat current) can be eliminated by means of a field dependent gauge transformation. As a result, the gauge equivalent Lax connection can easily be T-dualized, i.e. written in terms of the dual set of isometric coordinates. The T-dual Lax connection can be used for the derivation of infinitely many conserved charges in the T-dual model. Our construction implies that local (Noether) charges of the original model are mapped to non-local charges of the T-dual model and vice versa.

Keywords: Sigma Models, AdS-CFT Correspondence, Integrable Equations in Physics, Integrable Field Theories.

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Contents

1. Introduction and summary

Some of the recent remarkable advances in our understanding of $\mathcal{N} = 4$ superconformal Yang-Mills theory have been possible due to its integrability in the planar limit. This integrability was observed in the dilatation operator (or the spectrum of anomalous dimensions) at several leading orders in a weak coupling expansion and is expected to hold to all orders (see, e.g., [1] and references therein). Integrability is present also in the strong coupling limit of the $\mathcal{N} = 4$ SYM theory as described by the $\mathrm{AdS}_5 \times S^5$ superstring [2]. Indeed, the well-known classical integrability of the bosonic $\mathrm{AdS}_5 \times S^5$ sigma model was shown in [3] to extend to its κ-symmetric Green-Schwarz-type fermionic generalization [2]. In particular, an infinite number of conserved non-local charges for the classical superstring was found (see also [4]).[1] This integrability appears to extend also to one- and two-loop orders in the quantum string sigma model as implied by the computations of quantum string corrections to semi-classical string energies and their matching to predictions of the interpolating Bethe ansatz (see [9 − 11] and references there). The non-local conserved charges found on the string side appear to have a counterpart in planar gauge theory at weak coupling within the spin-chain formulation for the dilatation operator [12], but their direct interpretation in a field theoretic language is still missing.[2]

One expects that the integrability of the $\mathcal{N} = 4$ SYM theory should not only have important consequences for the spectrum of anomalous dimensions of gauge-invariant single trace operators (i.e. for the spectrum of energies of the *closed* $\mathrm{AdS}_5 \times S^5$ superstring) but also for other observables, e.g., for the structure of expectation values of certain Wilson

[1] Aspects related to involutivity of the charges were discussed in [5]. The existence of such charges was also verified in the pure spinor formulation of the $\mathrm{AdS}_5 \times S^5$ superstring [6] and shown to persist after quantization [7, 8].

[2] See also [13] for recent developments about the twistor approach to non-local symmetries.

loops. The dual AdS/CFT counterparts of the latter are partition functions of *open* $\mathrm{AdS}_5 \times S^5$ strings ending on certain contours at the boundary of AdS_5 [14]. The integrability can be used, e.g., for finding the corresponding minimal surfaces.[3] With a motivation to study quantum string corrections to Wilson loops, ref. [16] considered a special κ-symmetry gauge [17] in which the action of [2] written in the Poincaré coordinates of AdS_5 simplifies to

$$\begin{aligned} S[X,Y,\Theta] \;=\; -\frac{T}{2}\int_\Sigma \Big[& Y^2\,\eta_{ab}(\mathrm{d}X^a - 2\mathrm{i}\bar\Theta\Gamma^a\mathrm{d}\Theta)\wedge *(\mathrm{d}X^b - 2\mathrm{i}\bar\Theta\Gamma^b\mathrm{d}\Theta) + \\ & + \frac{1}{Y^2}\delta_{ij}\mathrm{d}Y^i\wedge *\mathrm{d}Y^j + 4\mathrm{i}\,\delta_{ij}\mathrm{d}Y^i\bar\Theta\wedge\Gamma^j\mathrm{d}\Theta\Big]. \end{aligned} \tag{1.1}$$

Here, $T=\frac{\sqrt{\lambda}}{2\pi}$ is the string tension, '$*$' is the Hodge star on the Minkowski world-sheet Σ, X^a are the four coordinates in the directions parallel to the boundary of AdS_5 with $(\eta_{ab}) = \mathrm{diag}(1,1,1,-1)$ $(a,b=1,\ldots,4)$ and Y^i are the six remaining coordinates with $Y^2 = \delta_{ij}Y^iY^j$ and $(\delta_{ij}) = \mathrm{diag}(1,\ldots,1)$ $(i,j=5,\ldots,10)$. Furthermore, Θ is a ten-dimensional Majorana-Weyl spinor and $\Gamma = (\Gamma^a,\Gamma^i)$ are the standard "flat" ten-dimensional Dirac matrices. It was observed in [16] that since the action (1.1) depends on the isometric coordinates X^a only through their derivatives, it can be simplified further if one trades X^a for the set of four dual two-dimensional scalars $\tilde X^a$, i.e. if one performs the formal T-duality transformation [18, 19] along X^a. Remarkably, the bosonic part of the resulting action has again an $\mathrm{AdS}_5\times S^5$ geometry (with $Y^2\mapsto 1/Y^2$, $X^a\mapsto\tilde X^a$) and its fermionic part becomes simply quadratic in Θ

$$\begin{aligned} \tilde S[\tilde X,Y,\Theta] \;=\; -\frac{T}{2}\int_\Sigma \bigg[& \frac{1}{Y^2}\Big(\eta_{ab}\mathrm{d}\tilde X^a\wedge *\mathrm{d}\tilde X^b + \delta_{ij}\mathrm{d}Y^i\wedge *\mathrm{d}Y^j\Big) + \\ & + 4\mathrm{i}\,\bar\Theta\Big(\eta_{ab}\mathrm{d}\tilde X^a\Gamma^b + \delta_{ij}\mathrm{d}Y^i\Gamma^j\Big)\wedge\mathrm{d}\Theta\bigg]. \end{aligned} \tag{1.2}$$

Like the bosonic part of the original action (1.1), the bosonic part of the T-dual action (1.2) has an $\mathrm{SO}(4,2)\times\mathrm{SO}(6)$ global symmetry, with the two $\mathrm{SO}(4,2)$ conformal groups acting, of course, on different (dual) sets of variables.[4]

Since the X^a-directions are non-compact, this T-duality is not an equivalence transformation on a two-dimensional cylinder, i.e. the transformed action is not appropriate as a starting point for the study of the closed string spectrum of the $\mathrm{AdS}_5\times S^5$ superstring. However, it may still be useful in the *open* string context which we will have in mind.

Indeed, this T-dual formulation appears to play an important role in the recently discovered connection between maximally helicity-violating (MHV) gluon scattering amplitudes [20] and special Wilson loops (defined on contours formed by light-like gluon

[3]See [15] for some applications of integrability to the computation of a wide class of Wilson loops.

[4]The full actions have only $\mathrm{SO}(3,1)\times\mathrm{SO}(6)$ as an obvious linearly realized symmetry. They are also invariant under the scaling transformations ($X^a\mapsto\ell X^a$, $Y\mapsto\ell^{-1}Y$ and $\tilde X^a\mapsto\ell^{-1}\tilde X^a$) (the invariance of the fermionic term can be seen by writing $Y^i = Yn^i$, $n^in_i=1$ and rescaling the fermions by $Y^{1/2}$-factor using that for Majorana-Weyl spinors $\bar\Theta\Gamma\Theta=0$). Other global symmetries of (1.1) are broken by the choice of the κ-symmetry gauge (i.e. they are present modulo a gauge transformation). Note also that quantum T-duality transformation produces also a dilaton term $\Phi=-2\ln(Y^2)$ which formally breaks the "dual" conformal symmetry. Here, we shall ignore it as all considerations in this paper will be classical.

momentum vectors) at strong [21, 22][5] and weak [24, 25] coupling. The classical SO(4, 2) conformal symmetry of the T-dual AdS geometry seems to have something to do with the mysterious "dual" conformal symmetry observed in the momentum-space integrands of loop integrals for planar gluon scattering amplitudes [24, 26]. From the point of view of the original $\mathrm{AdS}_5 \times S^5$ model, this dual conformal symmetry could be related to the presence of hidden symmetries associated with the integrability and, as such, it may correspond to the existence of non-local currents.

With a motivation to shed some light on these issues, it is essential to understand how the integrable structure emerges in the T-dual formulation and, in particular, how it translates from the original model to the T-dual one. Since T-duality maps classical solutions to classical solutions and also since the T-dual geometry is again[6] $\mathrm{AdS}_5 \times S^5$, it is reasonable to expect that the T-dual model is also integrable. One question then is how to map the Lax connection of the original model to the T-dual one, or how to map the non-local flat currents and the associated non-local conserved charges to the T-dual model. Some previous work on T-duality in the context of related integrable models appeared in [27–29] (see also [30, 31]).

Below we shall perform a first step in this direction by focusing for simplicity on the bosonic part of the model (1.1) and (1.2). After having presented the general setting in the next section, we shall then discuss the T-duality for two toy examples: the two-sphere S^2 and the two-dimensional anti-de Sitter space AdS_2. These two examples are different in nature as T-duality will be performed along a compact direction for S^2 and along a non-compact direction for AdS_2. On the other hand, they both exhibit some generic features useful to understand in view of our eventual aim — the sigma model on $\mathrm{AdS}_5 \times S^5$.

To perform the T-duality explicitly, one needs to express the T-dual coordinates in terms of the original ones. This relation is non-local; that makes it non-trivial to solve for the T-dual coordinates and to find the T-duality image of flat currents. Nevertheless, we demonstrate that it is possible to eliminate the explicit dependence of the flat currents on the T-duality direction coordinate by means of a finite field dependent gauge transformation. This yields a gauge equivalent Lax connection that depends only on the derivatives of the isometric coordinates. Our procedure is similar to the one in [27] (see also [29]). We then go on to discuss T-duality for general AdS geometries including AdS_5 case.

Having gauged away the explicit coordinate dependence, the T-duality on the flat currents of the AdS_5 sigma model can be easily implemented. This in turn allows us to find the T-dual flat currents representing the dual $\widetilde{\mathrm{SO}(4,2)}$ conformal symmetry. An application of this formalism would be the explicit construction of an infinite tower of conserved charges in both the original and the T-dual AdS_5 spaces and the investigation of their relation. It is likely, that these charges will only be well defined after a suitable regularization (cf. [21]).[7]

[5]The T-dual action (1.2) was used also for quantum one- and two-loop string computations in this context in [23, 11].

[6]We stress again that this is a formal T-duality transformation which maps locally the AdS sigma model into same sigma model; global issues are left aside in this work.

[7]We thank F. Alday for drawing our attention to this issue.

A natural next step would be an extension of the present analysis to the full superstring action (1.1). While the T-duality acts on the bosonic AdS coordinates in a relatively simple way, so that, e.g., the full conformal symmetry group present before the duality reappears after it, the situation is not as simple when one includes the fermions. The two dual actions (1.1) and (1.2) do not appear to be related by a local change of variables and renaming the fields (as it is at the bosonic level). In particular, some of the PSU$(2,2|4)$ superisometries present in (1.1) (those preserved by the κ-symmetry gauge fixing) are not manifest in the T-dual formalism. It would be important to understand whether the original symmetry group is still present but realized in some hidden, non-local way, as it is in the example of a much simpler bosonic sigma model on S^2 discussed in section 3.1 This may turn out to be helpful for a better understanding of Wilson loops in the T-dual AdS$_5$ space [21, 22].

2. Non-linear sigma models and non-local charges

2.1 Generalities

Non-linear sigma models. Let us consider the non-linear sigma model action

$$S[X] \;=\; -\frac{1}{2}\int_\Sigma g_{IJ}(X)\,\mathrm{d}X^I\wedge *\mathrm{d}X^J, \tag{2.1}$$

where $X:(\Sigma,h)\to(M,g)$ embeds a pseudo-Riemannian surface Σ with metric h into some d-dimensional pseudo-Riemannian manifold M (target space) with metric g; $I,J,\ldots=0,\ldots,d-1$. Below, we shall coordinatize Σ by t and x. Furthermore, we assume that there is some connected Lie group G acting on M by isometries. This implies that for any $V\in\mathfrak{g}:=\mathrm{Lie}(G)$ we have a corresponding vector field ξ_V,

$$\xi_V(f) \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\circ\exp(tV), \tag{2.2}$$

for some function f on M, such that (2.1) is invariant under

$$X^I \;\mapsto\; X^I+\xi^I_V, \qquad \text{with} \qquad \mathcal{L}_{\xi_V}g \;=\; 0. \tag{2.3}$$

Here, $\mathcal{L}_{\xi_V}$ denotes the Lie derivative along the vector field ξ_V. Put differently, for any $V\in\mathfrak{g}$, the vector field ξ_V is a Killing vector field of g.

As a short calculation reveals, the Noether current associated with the Killing vector field ξ_V takes the form[8]

$$\langle j,V\rangle \;=\; g_{IJ}(X)\mathrm{d}X^I\xi^J_V(X), \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ is a metric on $\mathfrak{g}$.

[8]Note that this current does not coincide with the one obtained from the stress tensor but differs by an improvement term.

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Flat currents and charges. Suppose now that in addition to being conserved, $\mathrm{d}{*j} = 0$, the current j also satisfies a flatness condition

$$\mathrm{d}j + j \wedge j \;=\; 0. \tag{2.5}$$

This occurs, for instance, when M is a symmetric space G/H, where G is the isometry group of M and H is the isotropy group of the action of G on M on some fixed $p \in M$. More generally, such flatness conditions arise when M is a coset space that admits a certain $\mathbb{Z}_m$-grading [32].

Given some conserved current j which is flat, then there are always two one-parameter families of flat currents J (Lax connection). Indeed, by considering general linear combinations of the form (with a, b being real numbers)

$$J \;=\; a\,j + b\,{*j}, \tag{2.6}$$

one observes that[9]

$$\mathrm{d}J + J \wedge J \;=\; (a^2 - a - b^2) j \wedge j \;\stackrel{!}{=}\; 0. \tag{2.7}$$

This can be solved by putting

$$J \;=\; J_{\lambda\pm} \qquad a \;=\; \frac{1}{2}[1 \pm \cosh(\lambda)] \qquad \text{and} \qquad b \;=\; \frac{1}{2}\sinh(\lambda)\,, \tag{2.8}$$

for $\lambda \in \mathbb{R}$. Note that in that case the zero curvature equation (2.6) encodes the equations of motion for the action (2.1).

As was shown in [33], taking the path-ordered exponential

$$W(t, x; t_0, x_0 | J_{\lambda\pm}) \;=\; P\exp\left(-\int_{\mathscr{C}} J_{\lambda\pm}\right) \tag{2.9}$$

along some unbounded spatial contour $\mathscr{C} \subset \Sigma$ at some fixed time, a given one-parameter family of flat currents $J_{\lambda\pm}$ always induces an infinite number of conserved non-local charges. To be concrete, for $J_{\lambda-}$, the quantity

$$Q_{\lambda-}(t) \;=\; \lim_{x\to\infty} W(t, x; t, -x | J_{\lambda-}) \;=\; 1 + \sum_{n=1}^{\infty} \lambda^n Q_n(t) \tag{2.10}$$

is conserved for all λ, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_n(t) \;=\; 0, \tag{2.11}$$

provided j has an appropriate fall-off at spatial infinity. The first two charges read as

$$\begin{aligned} Q_1(t) \;&=\; \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, j_0(t, x), \\ Q_2(t) \;&=\; -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}x\, j_1(t, x) + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{x} \mathrm{d}x'\, j_0(t, x) j_0(t, x'). \end{aligned} \tag{2.12}$$

The charge $Q_2(t)$ generates, via Poisson brackets, all the higher charges $Q_n(t)$ (for more details, see, e.g., the review in [34]).

[9] Here we used $**1 = 1$ and $*\alpha \wedge \beta + \alpha \wedge *\beta = 0$, for $\alpha, \beta \in \Omega^1\Sigma$.

2.2 AdS geometries

Let us now specify the case of AdS geometries that we shall be discussing later on. In particular, we shall derive the Killing vectors and the Noether currents in the Poincaré-patch parametrization we are interested in.

Setting. Consider the sigma model on the direct product $M = \mathrm{AdS}_p \times S^{d-p}$. We coordinatize M by (X^a, Y^i) with $a, b, \ldots = 1, \ldots, p-1$ and $i, j, \ldots = p, \ldots, d$, respectively, and equip it with the (conformally flat) metric

$$g \;=\; \frac{1}{Y^2}(\eta_{ab}\,\mathrm{d}X^a \otimes \mathrm{d}X^b + \delta_{ij}\,\mathrm{d}Y^i \otimes \mathrm{d}Y^j). \tag{2.13}$$

Here, we have introduced the following abbreviations: $Y^2 := \delta_{ij}Y^iY^j$, $(\delta_{ij}) = \mathrm{diag}(1, \ldots, 1)$ and $(\eta_{ab}) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p-r-1}, \underbrace{-1, \ldots, -1}_{r})$, where $r = 0$ for the Euclidean and $r = 1$ for the Minkowski AdS spaces.

Note that the metric g can be brought into its standard form by performing the following change of coordinates:

$$(X^a, Y^i) \;\mapsto\; (X^a, \hat{Y}^i) \;=\; (X^a, Y^{-1}Y^i). \tag{2.14a}$$

In these coordinates, g reads as

$$g \;=\; \underbrace{\frac{1}{Y^2}(\eta_{ab}\,\mathrm{d}X^a \otimes \mathrm{d}X^b + \mathrm{d}Y \otimes \mathrm{d}Y)}_{\mathrm{AdS}_p-\text{part}} + \underbrace{\delta_{ij}\,\mathrm{d}\hat{Y}^i \otimes \mathrm{d}\hat{Y}^j}_{S^{d-p}-\text{part}}, \tag{2.14b}$$

with $\delta_{ij}\hat{Y}^i\hat{Y}^j = 1$.

Below, when discussing T-duality, we shall choose to start with the AdS metric given in (2.13), i.e. we will do T-duality in the "opposite" direction compared to (1.1) and (1.2). This is, of course, simply a convention as the two choices are related by

$$(X^a, Y^i) \;\mapsto\; (X^a, \bar{Y}^i) \;=\; (X^a, Y^{-2}Y^i) \tag{2.15a}$$

and which results in

$$g \;=\; \eta_{ab}\,\bar{Y}^2\mathrm{d}X^a \otimes \mathrm{d}X^b + \frac{1}{\bar{Y}^2}\delta_{ij}\,\mathrm{d}\bar{Y}^i \otimes \mathrm{d}\bar{Y}^j, \tag{2.15b}$$

with $\bar{Y}^2 := \delta_{ij}\bar{Y}^i\bar{Y}^j$. The choice (2.13) will help us to simplify notation.

Killing vectors. Recalling the coset representations

$$\mathrm{AdS}_p \;\cong\; \mathrm{SO}(p-r, r+1)/\,\mathrm{SO}(p-r, r) \quad \text{and} \quad S^q \;\cong\; \mathrm{SO}(q+1)/\,\mathrm{SO}(q), \tag{2.16}$$

we observe that the isometry group of M is $G \cong \mathrm{SO}(p-r, r+1) \times \mathrm{SO}(d-p+1) \subset \mathrm{SO}(d, r+1)$. Hence, there are $\frac{1}{2}d(d+1) - p(d-p)$ Killing vectors which represent $\mathfrak{g} \cong \mathfrak{so}(p-r, r+1) \oplus \mathfrak{so}(d-p+1)$. In the (X^a, Y^j) parametrization of M, they are given by

$$\begin{aligned} \xi_{L_{ab}} &= X_a\partial_b - X_b\partial_a, & \xi_{M_{ij}} &= Y_i\partial_j - Y_j\partial_i, \\ \xi_{P_a} &= \partial_a, & \xi_D &= X^a\partial_a + Y^i\partial_i, \\ \xi_{K_a} &= (X^2 + Y^2)\partial_a - 2X_a(X^b\partial_b + Y^i\partial_i), \end{aligned} \tag{2.17}$$

where $X_a := \eta_{ab}X^b$ and $Y_i := \delta_{ij}Y^j$, $\partial_a := \partial/\partial X^a$ and $\partial_i := \partial/\partial Y^i$ and $X^2 := \eta_{ab}X^aX^b = X_aX^a$. The $\mathfrak{so}(p-r,r+1)$ and $\mathfrak{so}(d-p+1)$ Lie algebras are generated by L_{ab}, P_a, D, K_a and M_{ij}, respectively.

The non-vanishing commutators among the above vector fields are:

$$
\begin{aligned}
[\xi_{L_{ab}},\xi_{L_{cd}}] &= \eta_{bc}\xi_{L_{ad}} - \eta_{bd}\xi_{L_{ac}} - \eta_{ac}\xi_{L_{bd}} + \eta_{ad}\xi_{L_{bc}},\\
[\xi_{L_{ab}},\xi_{P_c}] &= \eta_{bc}\xi_{P_a} - \eta_{ac}\xi_{P_b}, & [\xi_{L_{ab}},\xi_{K_c}] &= \eta_{bc}\xi_{K_a} - \eta_{ac}\xi_{K_b},\\
[\xi_{P_a},\xi_D] &= \xi_{P_a}, & [\xi_{K_a},\xi_D] &= -\xi_{K_a},\\
[\xi_{P_a},\xi_{K_b}] &= 2\xi_{L_{ab}} - 2\eta_{ab}\xi_D,\\
[\xi_{M_{ij}},\xi_{M_{kl}}] &= \eta_{jk}\xi_{M_{il}} - \eta_{jl}\xi_{M_{ik}} - \eta_{ik}\xi_{M_{jl}} + \eta_{il}\xi_{M_{jk}}.
\end{aligned}
\tag{2.18}
$$

Flat currents. Knowing all the Killing vectors fields, we are now in the position to construct the associated Noether currents by using the formula (2.4). In the present situation, it reads as

$$
\langle j, V\rangle \;=\; \frac{1}{Y^2}(\eta_{ab}\mathrm{d}X^a\xi^b_V + \delta_{ij}\mathrm{d}Y^i\xi^j_V), \tag{2.19}
$$

where ξ_V represents any of the above Killing vectors fields for $V \in \mathfrak{g}$. Using their explicit expressions, we obtain

$$
\begin{aligned}
\langle j, L_{ab}\rangle &= \frac{1}{Y^2}(\mathrm{d}X_aX_b - \mathrm{d}X_bX_a), & \langle j, M_{ij}\rangle &= -\frac{1}{Y^2}(\mathrm{d}Y_iY_j - \mathrm{d}Y_jY_i),\\
\langle j, P_a\rangle &= \frac{1}{Y^2}\mathrm{d}X_a, & \langle j, D\rangle &= -\frac{2}{Y^2}(\mathrm{d}X^aX_a + \mathrm{d}Y^iY_i),\\
\langle j, K_a\rangle &= \frac{1}{Y^2}(X^2+Y^2)\mathrm{d}X_a - \frac{2}{Y^2}X_a(\mathrm{d}X^bX_b + \mathrm{d}Y^iY_i).
\end{aligned}
\tag{2.20}
$$

Then the current j, as constructed above, satisfies a flatness condition $\mathrm{d}j + j\wedge j = 0$. To see this, let us only exemplify the calculation for the L_{ab}-component of j, that is,

$$
\langle \mathrm{d}j + j\wedge j, L_{ab}\rangle \;=\; 0. \tag{2.21}
$$

The others are verified in a similar manner.

To compute the projection of $\mathrm{d}j+j\wedge j$ onto the rotation generator L_{ab}, one first realizes that one needs to consider also the projections onto P_a and K_a, respectively, which follows upon inspecting the above commutation relations of the Killing vectors. Then one finds

$$
\langle \mathrm{d}j, L_{ab}\rangle \;=\; \mathrm{d}\langle j, L_{ab}\rangle \;=\; -\frac{2}{Y^4}\left[Y^i\mathrm{d}Y_i\wedge(\mathrm{d}X_aX_b - \mathrm{d}X_bX_a) + Y^2\mathrm{d}X_a\wedge\mathrm{d}X_b\right]. \tag{2.22a}
$$

Similarly, one obtains ($\eta_{ac}\eta^{cb} = \delta_a{}^b$)

$$
\begin{aligned}
\langle j\wedge j, L_{ab}\rangle \;&=\; 2\eta^{cd}\langle j, L_{ac}\rangle\wedge\langle j, L_{db}\rangle + \langle j, P_a\rangle\wedge\langle j, K_b\rangle - \langle j, P_b\rangle\wedge\langle j, K_a\rangle\\
&=\; \frac{2}{Y^4}\left[Y^i\mathrm{d}Y_i\wedge(\mathrm{d}X_aX_b - \mathrm{d}X_bX_a) + Y^2\mathrm{d}X_a\wedge\mathrm{d}X_b\right],
\end{aligned}
\tag{2.22b}
$$

so that the combination of the two expressions is indeed zero.

Following the algorithm presented in section 2.1, one can then construct two one-parameter families of flat currents and hence, an infinite number of conserved non-local charges. This familiar result is, of course, a consequence of $M = \mathrm{AdS}_p \times S^{d-p}$ being a symmetric space.

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Remark. In the following, we shall choose $r = 0$ for notational simplicity, but all the relations derived below are true also for a generic space-time signature.

3. T-duality

Subject of this section is the discussion of certain integrable non-linear sigma models and the derivation of their dual cousins by T-duality along certain isometries. In particular, we will find the flat currents of T-dual models.

3.1 T-duality and flat currents for S^2

Setting. Before turning our attention to AdS geometries, we will consider the two-sphere S^2 and perform T-duality along the compact U(1) isometry cycle. This example differs then from our main focus, i.e. the AdS_p space, where T-duality is performed along non-compact isometric directions, but it serves to illustrate in a simple setting a point that will apply also to AdS_p geometries — that the local symmetries of the original model become hidden and are realized non-locally in the T-dual theory.

The S^2 sigma model action reads

$$S[\Phi,\Theta] \;=\; -\frac{1}{2}\int_\Sigma \big[\mathrm{d}\Theta\wedge *\mathrm{d}\Theta + \sin^2(\Theta)\,\mathrm{d}\Phi\wedge *\mathrm{d}\Phi\big]. \tag{3.1}$$

Following the procedure in section 2, the Noether currents are found to be

$$\begin{aligned} j_1 &= \frac{1}{\sqrt{2}}\,[\sin(\Phi)\,\mathrm{d}\Theta + \sin(\Theta)\cos(\Theta)\cos(\Phi)\,\mathrm{d}\Phi],\\ j_2 &= \frac{1}{\sqrt{2}}\,[\cos(\Phi)\,\mathrm{d}\Theta - \sin(\Theta)\cos(\Theta)\sin(\Phi)\,\mathrm{d}\Phi],\\ j_3 &= \frac{1}{\sqrt{2}}\sin^2(\Theta)\,\mathrm{d}\Phi. \end{aligned} \tag{3.2}$$

Besides being conserved, these currents are also flat

$$\mathrm{d}j_A + f_A{}^{BC} j_B\wedge j_C \;=\; 0, \qquad \text{with} \qquad f_A{}^{BC} \;=\; -\frac{1}{\sqrt{2}}k_{AD}\epsilon^{DBC} \tag{3.3}$$

for $A, B, \ldots = 1, 2, 3$. Here, ϵ^{ABC} is totally antisymmetric with $\epsilon^{123} = 1$ and the Cartan-Killing form k is given by $k = (k_{AB}) = \mathrm{diag}(-1,-1,-1)$. Therefore, we find two one-parameter families of flat currents[10]

$$J \;=\; aj + b*j, \quad \text{with} \quad a \;=\; \tfrac{1}{2}[1\pm\cosh(\lambda)] \quad \text{and} \quad b \;=\; \tfrac{1}{2}\sinh(\lambda), \tag{3.4}$$

i.e.

$$\mathrm{d}J_A + f_A{}^{BC} J_B\wedge J_C \;=\; 0, \qquad \text{with} \qquad J_A \;=\; aj_A + b*j_A, \tag{3.5}$$

from which infinitely many non-local conserved charges may be derived.

[10] Here and in the following we shall suppress the subscript '$\lambda\pm$' in (2.8) and write J instead of $J_{\lambda\pm}$.

T-duality. We will now perform T-duality along the Φ-direction. As usual, this may be implemented by starting with the first-order action [18, 19]

$$S[\Theta, A, \tilde{\Phi}] \; = \; -\frac{1}{2}\int_\Sigma \Big[\mathrm{d}\Theta \wedge *\mathrm{d}\Theta + \sin^2(\Theta) A \wedge *A + 2\,\tilde{\Phi}\,\mathrm{d}A\Big], \tag{3.6}$$

where the one-form A is an Abelian gauge potential and the field $\tilde{\Phi}$ (to be later interpreted as T-dual to Φ) plays the role of a Lagrange multiplier for the field strength $F = \mathrm{d}A$.[11] By integrating out $\tilde{\Phi}$, we see that the gauge potential A is pure gauge

$$A \; = \; \mathrm{d}\Phi, \tag{3.7}$$

and upon substitution into eq. (3.6), we recover the original action (3.1). On the other hand, the variation with respect to A yields

$$A \; = \; \frac{1}{\sin^2(\Theta)} *\mathrm{d}\tilde{\Phi}, \tag{3.8}$$

which implies the relation

$$\mathrm{d}\tilde{\Phi} \; = \; \sin^2(\Theta) *\mathrm{d}\Phi \tag{3.9}$$

between the original field Φ and its T-dual $\tilde{\Phi}$. In terms of $\tilde{\Phi}$, the T-dual action is given by

$$\tilde{S}[\tilde{\Phi}, \Theta] \; = \; -\frac{1}{2}\int_\Sigma \left[\mathrm{d}\Theta \wedge *\mathrm{d}\Theta + \frac{1}{\sin^2(\Theta)}\,\mathrm{d}\tilde{\Phi} \wedge *\mathrm{d}\tilde{\Phi}\right]. \tag{3.10}$$

The fact that $\Theta = 0, \pi$ are fixed points of the $\mathrm{U}(1)$ isometry manifests in a T-dual geometry with singularities at those points.

While the original action was SO(3) invariant, the manifest symmetry of the T-dual action is simply $\mathrm{U}(1)$ shifts of $\tilde{\Phi}$. Therefore, if we consider only the Noether currents of the T-dual model, we will never be able to see the full SO(3) symmetry group which the T-dual model should also admit, given the two sets of equations are equivalent. This non-Abelian symmetry group of the T-dual model, which we shall call "T-dual symmetry group" and denote by $\widetilde{\mathrm{SO}(3)}$, will be hidden and, in addition, realized non-locally.

T-dual symmetry group. To find this T-dual symmetry group, let us go back to the flat currents of the original model. The T-duality transformation (3.9) cannot be directly performed on the currents since they depend not only on $\mathrm{d}\Phi$ but also explicitly on the coordinate Φ. Note, however, that flat currents $\mathrm{d}J + J \wedge J = 0$ are unique only up to a G-gauge transformation ($G = \mathrm{SO}(3)$ in the present case)

$$J \;\mapsto\; J' \; = \; g^{-1}Jg + g^{-1}\mathrm{d}g, \tag{3.11}$$

where $g \in G$: J' is again flat. Then there exists an element $g : \Sigma \to \mathrm{SO}(3)$ which transforms the original currents into a new gauge-equivalent set of currents that depend on Φ only through its derivatives.

[11] In principle, the gauge potential A might have non-trivial holonomies around non-contractible loops. This can be avoided by requiring $\tilde{\Phi}$ to have the appropriate periodicity. Since our main interest in this paper will be T-duality along non-compact directions, we will not discuss this issue further (see [19]).

To construct such g, let us work in the fundamental representation of the group SO(3). The generators of SO(3) obey

$$[T_A, T_B] \;=\; f_{AB}{}^C T_C, \qquad \text{with} \qquad f_{AB}{}^C \;=\; -\frac{1}{\sqrt{2}}\epsilon_{ABD}k^{DC}, \tag{3.12}$$

where $k = (k_{AB}) = \mathrm{diag}(-1,-1,-1)$ is as before; note that $\epsilon_{123} = -1$. They can be chosen as

$$\begin{aligned} T_1 \;&=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad T_2 \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ T_3 \;&=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.13}$$

Then the current $J = J_A T^A = J_A k^{AB} T_B$ takes the following matrix form:

$$J \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -J_3 & J_2 \\ J_3 & 0 & -J_1 \\ -J_2 & J_1 & 0 \end{pmatrix}. \tag{3.14}$$

One can check that the choice of

$$g \;=\; \begin{pmatrix} \cos(\Phi) & \sin(\Phi) & 0 \\ -\sin(\Phi) & \cos(\Phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.15}$$

yields the desired result:

$$J' \;=\; g^{-1}Jg + g^{-1}\mathrm{d}g \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -J'_3 & J'_2 \\ J'_3 & 0 & -J'_1 \\ -J'_2 & J'_1 & 0 \end{pmatrix} \tag{3.16}$$

with (recall that $a^2 - a - b^2 = 0$, see (3.4))

$$\begin{aligned} J'_1 \;&=\; \frac{1}{\sqrt{2}}\sin(\Theta)\cos(\Theta)\,(a\,\mathrm{d}\Phi + b\,{*\mathrm{d}\Phi}), \\ J'_2 \;&=\; \frac{1}{\sqrt{2}}(a\,\mathrm{d}\Theta + b\,{*\mathrm{d}\Theta}), \\ J'_3 \;&=\; \frac{1}{\sqrt{2}}\sin^2(\Theta)\,(a\,\mathrm{d}\Phi + b\,{*\mathrm{d}\Phi}) - \sqrt{2}\,\mathrm{d}\Phi. \end{aligned} \tag{3.17}$$

Using (3.9), we then find the currents of the T-dual model: $\tilde{J}_A := J'_A(\Phi \mapsto \tilde{\Phi})$, i.e.

$$\begin{aligned} \tilde{J}_1 \;&=\; \frac{1}{\sqrt{2}}\cot(\Theta)\,(a\,{*\mathrm{d}\tilde{\Phi}} + b\,\mathrm{d}\tilde{\Phi}), \\ \tilde{J}_2 \;&=\; \frac{1}{\sqrt{2}}(a\,\mathrm{d}\Theta + b\,{*\mathrm{d}\Theta}), \\ \tilde{J}_3 \;&=\; \frac{1}{\sqrt{2}}(a\,{*\mathrm{d}\tilde{\Phi}} + b\,\mathrm{d}\tilde{\Phi}) - \frac{\sqrt{2}}{\sin^2(\Theta)}{*\mathrm{d}\tilde{\Phi}}. \end{aligned} \tag{3.18}$$

They are again flat, $\mathrm{d}\tilde{J} + \tilde{J} \wedge \tilde{J} = 0$, since the relation $\mathrm{d}\tilde{\Phi} = \sin^2(\Theta) * \mathrm{d}\Phi$ holds on-shell.

Proceeding as in section 2, we get then infinitely many conserved non-local charges. However, contrary to the original model we started with, the lowest order charges[12] are non-local and do not correspond to Noether charges. In this sense, local charges are mapped into non-local ones via T-duality. Conversely, the $\mathrm{U}(1)$ Noether charge of the T-dual model is mapped into a non-local charge in the original model.

The AdS geometries discussed below are quite different in this respect as they allow for a maximal set of Noether charges in both the original and the T-dual model. This means that both the original sigma model action on the AdS_p space and the one obtained after performing T-duality along all the isometric directions, possess an $\mathrm{SO}(p-1,1)$ symmetry group associated with the Noether charges.

3.2 T-duality and flat currents for AdS_2

Next, let us consider the simplest AdS example: AdS_2. The gauge transformation of the flat current constructed for this case will turn out to be the building block for all higher-dimensional AdS cases.

Setting. As before, we choose the conformally flat metric on AdS_2

$$S[X,Y] \;=\; -\frac{1}{2}\int_\Sigma \frac{1}{Y^2}\left(\mathrm{d}X \wedge *\mathrm{d}X + \mathrm{d}Y \wedge *\mathrm{d}Y\right). \tag{3.19}$$

The Noether currents have been derived in section 2, and we repeat them here:

$$\begin{aligned} j_1 \;&=\; -\frac{1}{2\sqrt{2}Y^2}\left\{\left[1+(X^2-Y^2)\right]\mathrm{d}X + 2XY\mathrm{d}Y\right\},\\ j_2 \;&=\; \frac{1}{2\sqrt{2}Y^2}\left\{\left[1-(X^2-Y^2)\right]\mathrm{d}X - 2XY\mathrm{d}Y\right\},\\ j_3 \;&=\; -\frac{1}{\sqrt{2}Y^2}\left(X\mathrm{d}X + Y\mathrm{d}Y\right). \end{aligned} \tag{3.20}$$

Here, we have taken certain linear combinations of the translation and special conformal currents and also introduced different normalization constants. As a result, these currents are flat, i.e. they obey

$$\mathrm{d}j_A + f_A{}^{BC} j_B \wedge j_C \;=\; 0, \qquad \text{with} \qquad f_A{}^{BC} \;=\; \frac{1}{\sqrt{2}} k_{AD}\epsilon^{DBC}, \tag{3.21}$$

where the Cartan-Killing form k is given by $k = (k_{AB}) = \mathrm{diag}(-1,1,1)$. Again, we may introduce a family of flat currents according to

$$J \;=\; a\, j + b * j. \tag{3.22}$$

[12]Here, we are considering the solution $a = \frac{1}{2}[1-\cosh(\lambda)]$ and $b = \frac{1}{2}\sinh(\lambda)$ and expand around $\lambda = 0$.

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T-duality. Performing T-duality along the X-direction by repeating the steps that led to the T-dual action in the S^2 example, we find

$$\mathrm{d}\tilde{X} \;=\; \frac{1}{Y^2}{*\mathrm{d}X} \tag{3.23}$$

and hence,

$$\tilde{S}[\tilde{X},Y] \;=\; -\frac{1}{2}\int_\Sigma \left(Y^2\mathrm{d}\tilde{X}\wedge *\mathrm{d}\tilde{X} + \frac{1}{Y^2}\mathrm{d}Y\wedge *\mathrm{d}Y\right). \tag{3.24}$$

This is again an AdS_2 sigma model. Getting back after T-duality the space one has started with is a special feature of AdS geometries (cf. S^2 example).[13]

Therefore, the T-dual model also exhibits an SO(2,1) isometry group. Upon constructing the T-dual Noether currents (in the T-dual coordinates), one may derive flat currents in the T-dual model which induce infinitely many conserved non-local charges. As in the original model, the lowest order charges are the Noether charges. This time, these are, of course, the Noether charges for the T-dual model in the T-dual coordinates.

T-dual symmetry group. Let us now discuss a similar procedure as in the two-sphere example, that is, let us show that there exists a transformation mediated by $g:\Sigma\to \mathrm{SO}(2,1)$ which maps the original currents into a new set of gauge equivalent currents that depend on X only through its derivatives.

As before, we shall work in the fundamental representation. The generators obey

$$[T_A,T_B] \;=\; f_{AB}{}^C T_C, \qquad \text{with} \qquad f_{AB}{}^C \;=\; \frac{1}{\sqrt{2}}\epsilon_{ABD}k^{DC}, \tag{3.25}$$

where $k=(k_{AB})=\mathrm{diag}(-1,1,1)$ is the Cartan-Killing form; note that $\epsilon_{123}=-1$.

The generators can be chosen as

$$\begin{aligned} T_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\\ T_3 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}. \end{aligned} \tag{3.26}$$

so that $J=J_AT^A=J_Ak^{AB}T_B$ is given by

$$J \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -J_1 & -J_2\\ J_1 & 0 & -J_3\\ -J_2 & -J_3 & 0\end{pmatrix}. \tag{3.27}$$

[13] In fact, any geometry with a metric of the form

$$g \;=\; f(Y)\,\mathrm{d}Y\otimes\mathrm{d}Y + h(Y)\,\mathrm{d}X\otimes\mathrm{d}X$$

goes back into itself after performing a T-duality along isometric direction X provided and $Y=Y(\tilde{Y})$ with $[Y'(\tilde{Y})]^2 \;=\; f(\tilde{Y})/f(Y(\tilde{Y}))$ and $h(\tilde{Y}) \;=\; [h(Y(\tilde{Y}))]^{-1}$. A simple example is $f=1$, $h(Y)=\exp(Y)$ and $Y=-\tilde{Y}$.

Then a short calculation reveals that

$$g = \begin{pmatrix} X & 1 & -X \\ 1-\frac{1}{2}X^2 & -X & \frac{1}{2}X^2 \\ \frac{1}{2}X^2 & X & -1-\frac{1}{2}X^2 \end{pmatrix} \in \mathrm{SO}(2,1) \tag{3.28}$$

gauges away the explicit X-dependence of the currents (3.20) and (3.22). The components of $J' = g^{-1}Jg + g^{-1}\mathrm{d}g$ are found to be

$$\begin{aligned} J_1' &= \frac{1}{2\sqrt{2}Y^2}(1-Y^2)(a\,\mathrm{d}X + b\,{*\mathrm{d}X}) + \sqrt{2}\,\mathrm{d}X, \\ J_2' &= \frac{1}{\sqrt{2}Y}(a\,\mathrm{d}Y + b\,{*\mathrm{d}Y}), \\ J_3' &= -\frac{1}{2\sqrt{2}Y^2}(1+Y^2)(a\,\mathrm{d}X + b\,{*\mathrm{d}X}) + \sqrt{2}\,\mathrm{d}X. \end{aligned} \tag{3.29}$$

The T-dual currents

$$\tilde{J}_A(Y,\tilde{X}) = J_A'(Y, X(\tilde{X})) \tag{3.30}$$

are then given by

$$\begin{aligned} \tilde{J}_1 &= \frac{1}{2\sqrt{2}}(1-Y^2)(a\,{*\mathrm{d}\tilde{X}} + b\,\mathrm{d}\tilde{X}) + \sqrt{2}Y^2\,{*\mathrm{d}\tilde{X}}, \\ \tilde{J}_2 &= \frac{1}{\sqrt{2}Y}(a\,\mathrm{d}Y + b\,{*\mathrm{d}Y}), \\ \tilde{J}_3 &= -\frac{1}{2\sqrt{2}}(1+Y^2)(a\,{*\mathrm{d}\tilde{X}} + b\,\mathrm{d}\tilde{X}) + \sqrt{2}Y^2\,{*\mathrm{d}\tilde{X}}, \end{aligned} \tag{3.31}$$

and are again flat (recall that $a^2 - a - b^2 = 0$).

Note that the lowest order charges obtained by expanding the path-ordered exponential (2.9) around $\lambda = 0$ are not the Noether charges for the T-dual model. The latter follow from the T-dual flat currents $\hat{J} = \hat{a}\hat{j} + \hat{b}\,{*\hat{j}}$, where $\hat{j}$ is now given by

$$\begin{aligned} \hat{j}_1 &= -\frac{1}{2\sqrt{2}}\big\{\big[Y^2 - (1-\tilde{X}^2Y^2)\big]\mathrm{d}\tilde{X} - \frac{2\tilde{X}}{Y}\mathrm{d}Y\big\}, \\ \hat{j}_2 &= \frac{1}{2\sqrt{2}}\big\{\big[Y^2 + (1-\tilde{X}^2Y^2)\big]\mathrm{d}\tilde{X} + \frac{2\tilde{X}}{Y}\mathrm{d}Y\big\}, \\ \hat{j}_3 &= -\frac{1}{\sqrt{2}}\big(Y^2\tilde{X}\mathrm{d}\tilde{X} - \frac{1}{Y}\mathrm{d}Y\big) \end{aligned} \tag{3.32}$$

and $\hat{a}^2 - \hat{a} - \hat{b}^2 = 0$, with

$$\hat{a} = \frac{1}{2}[1 \pm \cosh(\hat{\lambda})] \qquad \text{and} \qquad \hat{b} = \frac{1}{2}\sinh(\hat{\lambda}). \tag{3.33}$$

By the above reasoning, the $\tilde{X}$-dependence of $\hat{J}$ can be gauged away to get

$$\begin{aligned} \hat{J}_1' &= -\frac{1}{2\sqrt{2}}(1-Y^2)(\hat{a}\,\mathrm{d}\tilde{X} + \hat{b}\,{*\mathrm{d}\tilde{X}}) + \sqrt{2}\,\mathrm{d}\tilde{X}, \\ \hat{J}_2' &= \frac{1}{\sqrt{2}Y}(\hat{a}\,\mathrm{d}Y + \hat{b}\,{*\mathrm{d}Y}), \\ \hat{J}_3' &= -\frac{1}{2\sqrt{2}}(1+Y^2)(\hat{a}\,\mathrm{d}\tilde{X} + \hat{b}\,{*\mathrm{d}\tilde{X}}) + \sqrt{2}\,\mathrm{d}\tilde{X}. \end{aligned} \tag{3.34}$$

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In this sense, we get the following relation:

$$\begin{array}{ccc} \underline{\textbf{Original model}} & & \underline{\textbf{T-dual model}} \\ \text{local } \mathrm{SO}(2,1) & \Longrightarrow & \text{non-local } \widetilde{\mathrm{SO}(2,1)}, \\ \text{non-local } \widetilde{\mathrm{SO}(2,1)} & \Longleftarrow & \text{local } \mathrm{SO}(2,1). \end{array}$$

A similar interchanging of Noether and non-local charges was observed earlier in the context of type II strings on a pp-wave background in [28].

3.3 T-duality and flat currents for AdS_5

Let us now turn to higher-dimensional AdS geometries. For concreteness, we shall stick to AdS_5 but the derivations presented below can straightforwardly be extended to any dimension by means of a recursive procedure. As already indicated, the transformation for the AdS_2 space derived above will form a basic building block.

Setting. As before, we shall start with the Euclidean ($r = 0$) AdS metric in the conformally flat form[14]

$$g \;=\; \frac{1}{Y^2}(\,\mathrm{d}X_a \otimes \mathrm{d}X_a + \mathrm{d}Y \otimes \mathrm{d}Y), \tag{3.35}$$

The Noether currents for the corresponding sigma model are

$$\begin{aligned}
j_1 &= -\frac{1}{2\sqrt{2}Y^2}\left\{\left[1 + X_1^2 - (X_{(1)}\cdot X_{(1)} + Y^2)\right]\mathrm{d}X_1 + 2X_1(X_{(1)}\cdot \mathrm{d}X_{(1)} + Y\mathrm{d}Y)\right\},\\
j_2 &= \frac{1}{2\sqrt{2}Y^2}\left\{\left[1 - X_1^2 + (X_{(1)}\cdot X_{(1)} + Y^2)\right]\mathrm{d}X_1 - 2X_1(X_{(1)}\cdot \mathrm{d}X_{(1)} + Y\mathrm{d}Y)\right\},\\
j_3 &= -\frac{1}{\sqrt{2}Y^2}\left(X_1\mathrm{d}X_1 + X_{(1)}\cdot \mathrm{d}X_{(1)} + Y\mathrm{d}Y\right),\\
j_4 &= -\frac{1}{2\sqrt{2}Y^2}\left\{\left[1 + X_2^2 - (X_{(2)}\cdot X_{(2)} + Y^2)\right]\mathrm{d}X_2 + 2X_2(X_{(2)}\cdot \mathrm{d}X_{(2)} + Y\mathrm{d}Y)\right\},\\
j_5 &= \frac{1}{2\sqrt{2}Y^2}\left\{\left[1 - X_2^2 + (X_{(2)}\cdot X_{(2)} + Y^2)\right]\mathrm{d}X_2 - 2X_2(X_{(2)}\cdot \mathrm{d}X_{(2)} + Y\mathrm{d}Y)\right\},\\
j_6 &= -\frac{1}{\sqrt{2}Y^2}\left(X_1\mathrm{d}X_2 - X_2\mathrm{d}X_1\right),\\
j_7 &= -\frac{1}{2\sqrt{2}Y^2}\left\{\left[1 + X_3^2 - (X_{(3)}\cdot X_{(3)} + Y^2)\right]\mathrm{d}X_3 + 2X_3(X_{(3)}\cdot \mathrm{d}X_{(3)} + Y\mathrm{d}Y)\right\},\\
j_8 &= \frac{1}{2\sqrt{2}Y^2}\left\{\left[1 - X_3^2 + (X_{(3)}\cdot X_{(3)} + Y^2)\right]\mathrm{d}X_3 - 2X_3(X_{(3)}\cdot \mathrm{d}X_{(3)} + Y\mathrm{d}Y)\right\},\\
j_9 &= -\frac{1}{\sqrt{2}Y^2}\left(X_1\mathrm{d}X_3 - X_3\mathrm{d}X_1\right),\\
j_{10} &= -\frac{1}{\sqrt{2}Y^2}\left(X_2\mathrm{d}X_3 - X_3\mathrm{d}X_2\right),\\
j_{11} &= -\frac{1}{2\sqrt{2}Y^2}\left\{\left[1 + X_4^2 - (X_{(4)}\cdot X_{(4)} + Y^2)\right]\mathrm{d}X_4 + 2X_4(X_{(4)}\cdot \mathrm{d}X_{(4)} + Y\mathrm{d}Y)\right\},\\
j_{12} &= \frac{1}{2\sqrt{2}Y^2}\left\{\left[1 - X_4^2 + (X_{(4)}\cdot X_{(4)} + Y^2)\right]\mathrm{d}X_4 - 2X_4(X_{(4)}\cdot \mathrm{d}X_{(4)} + Y\mathrm{d}Y)\right\},
\end{aligned}$$

[14] The indices $a, b = 1, \ldots, 4$ here are contracted with δ_{ab}.

$$
\begin{aligned}
j_{13} &= -\frac{1}{\sqrt{2}Y^2}\,(X_1 \mathrm{d}X_4 - X_4 \mathrm{d}X_1), \\
j_{14} &= -\frac{1}{\sqrt{2}Y^2}\,(X_2 \mathrm{d}X_4 - X_4 \mathrm{d}X_2), \\
j_{15} &= -\frac{1}{\sqrt{2}Y^2}\,(X_3 \mathrm{d}X_4 - X_4 \mathrm{d}X_3),
\end{aligned} \tag{3.36}
$$

where we have introduced the abbreviation

$$
X_{(a)} \;=\; (\cdots, X_{a-1}, X_{a+1}, \ldots). \tag{3.37}
$$

We shall also use $X_{(0)} := (X_1, \ldots, X_4)$, in the sequel. The '$\cdot$' refers to the (Euclidean) scalar product, e.g. $X_{(1)} \cdot X_{(1)} = X_2^2 + X_3^2 + X_4^2$, etc. Note that we have again taken certain linear combinations of the translation and special conformal currents. The AdS_2 case can be recovered for $X_{2,3,4} = 0$, AdS_3 for $X_{3,4} = 0$ and AdS_4 for $X_4 = 0$, respectively. In addition, the above currents have been grouped according to their appearance: $j_{1,2,3}$ are the AdS_2 currents, $j_{1,\ldots,6}$ the AdS_3 currents, $j_{1,\ldots,10}$ the AdS_4 currents and $j_{1,\ldots,15}$ the AdS_5 currents (after putting to zero the appropriate X_a-coordinates).

These currents are flat

$$
\mathrm{d}j_A + f_A{}^{BC} j_B \wedge j_C \;=\; 0, \qquad \text{for} \qquad A, B, \ldots = 1, \ldots, 15, \tag{3.38}
$$

where $f_{AB}{}^C$ are the structure constants of $\mathfrak{so}(5,1)$. Again, we may introduce the one-parameter family of flat currents $J = aj + b*j$, with $a^2 - a - b^2 = 0$.

T-duality. Our main observation is that, as in the AdS_2 example, one is able to gauge away the X_a-dependence by means of a field dependent gauge transformation. This makes it possible to perform T-duality along the X_a-directions in a straightforward way.

As before, we work in the fundamental representation of $\mathrm{SO}(5,1)$. Let us denote the generators by T_A. The current $J = J_A T^A = J_A k^{AB} T_B$, with $J_A = a j_A + b * j_A$ and j_A given in (3.36), might be represented as[15]

$$
J \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -J_{15} & -J_{14} & -J_{13} & -J_{11} & -J_{12} \\
J_{15} & 0 & -J_{10} & -J_9 & -J_7 & -J_8 \\
J_{14} & J_{10} & 0 & -J_6 & -J_4 & -J_5 \\
J_{13} & J_9 & J_6 & 0 & -J_1 & -J_2 \\
J_{11} & J_7 & J_4 & J_1 & 0 & -J_3 \\
-J_{12} & -J_8 & -J_5 & -J_2 & -J_3 & 0
\end{pmatrix}. \tag{3.39}
$$

Here, the Cartan-Killing form is given by

$$
(k_{AB}) \;=\; \mathrm{diag}(-1, 1, 1, -1, 1, -1, -1, 1, -1, -1, -1, 1, -1, -1, -1). \tag{3.40}
$$

[15]Recall that the generators in the fundamental representation are given by

$$
(t_{\alpha\beta})^{\gamma\delta} \;=\; \delta_\alpha{}^\gamma \delta_\beta{}^\delta - \delta_\beta{}^\gamma \delta_\alpha{}^\delta, \qquad \text{with} \qquad \alpha, \beta, \ldots \;=\; 1, \ldots, 6.
$$

Upon relabeling the set $\{t_{\alpha\beta}\} \mapsto \{T_A\}$, one obtains the present choice of parametrization.

Notice that the lower right 3×3-block in eq. (3.39) represents the AdS_2 case, the 4×4-block the AdS_3 case, etc.

The key idea is to gauge away the explicit dependence on the X_a-coordinates recursively, i.e. first X_1, then X_2, etc. Following this algorithm, one finds that the required gauge transformation matrix is

$$g = g_1 g_2 g_3 g_4, \tag{3.41a}$$

with

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 & 1 & -X_1 \\ 0 & 0 & 0 & 1-\frac{1}{2}X_1^2 & -X_1 & \frac{1}{2}X_1^2 \\ 0 & 0 & 0 & \frac{1}{2}X_1^2 & X_1 & -1-\frac{1}{2}X_1^2 \end{pmatrix}, \tag{3.41b}$$

$$g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_2 & 1 & -\mathrm{i}X_2 & 0 \\ 0 & 0 & 1-\frac{1}{2}X_2^2 & -X_2 & \frac{\mathrm{i}}{2}X_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\frac{1}{2}X_2^2 & -X_2 & \mathrm{i}+\frac{\mathrm{i}}{2}X_2^2 & 0 \end{pmatrix}, \tag{3.41c}$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_3 & 1 & -\mathrm{i}X_3 & 0 & 0 \\ 0 & 1-\frac{1}{2}X_3^2 & -X_3 & \frac{\mathrm{i}}{2}X_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{\mathrm{i}}{2}X_3^2 & \mathrm{i}X_3 & 1+\frac{1}{2}X_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.41d}$$

$$g_4 = \begin{pmatrix} X_4 & 1 & -\mathrm{i}X_4 & 0 & 0 & 0 \\ 1-\frac{1}{2}X_4^2 & -X_4 & \frac{\mathrm{i}}{2}X_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{\mathrm{i}}{2}X_4^2 & \mathrm{i}X_4 & 1+\frac{1}{2}X_4^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.41e}$$

i.e.

$$g = \begin{pmatrix} X_4 & 1 & -\mathrm{i}X_4 & 0 & 0 & 0 \\ X_3 & 0 & -\mathrm{i}X_3 & 1 & 0 & 0 \\ X_2 & 0 & -\mathrm{i}X_2 & 0 & 1 & 0 \\ X_1 & 0 & -\mathrm{i}X_1 & 0 & 0 & -\mathrm{i} \\ 1-\frac{1}{2}X_{(0)}\cdot X_{(0)} & -X_4 & \frac{\mathrm{i}}{2}X_{(0)}\cdot X_{(0)} & -X_3 & -X_2 & \mathrm{i}X_1 \\ \frac{1}{2}X_{(0)}\cdot X_{(0)} & X_4 & -\mathrm{i}-\frac{\mathrm{i}}{2}X_{(0)}\cdot X_{(0)} & X_3 & X_2 & -\mathrm{i}X_1 \end{pmatrix}. \tag{3.41f}$$

As one can check, $g_1 \in \mathrm{SO}(5,1)$ while $g_{2,3,4}$ are elements of the complexified gauge group $SO_{\mathbb{C}}(5,1)$ (here, $\mathrm{i} := \sqrt{-1}$). Hence, $g \in SO_{\mathbb{C}}(5,1)$.

Let us stress that the appearance of the complexified gauge algebra $\mathfrak{so}_{\mathbb{C}}(5,1)$ and gauge group $SO_{\mathbb{C}}(5,1)$ is merely a consequence of the fact that we have chosen to start with the Euclidean AdS_5 space. Indeed, if one instead starts with Minkowski (or Kleinian, i.e. split signature) AdS_5 space, all the four g_m-matrices ($m = 1, \ldots, 4$) live in $\mathrm{SO}(4,2)$ (or $\mathrm{SO}(3,3)$); this can also be seen by making a suitable Wick rotation.[16]

It remains to write down the gauge transformed currents,

$$J' \;=\; g^{-1}Jg + g^{-1}\mathrm{d}g. \tag{3.42}$$

Their non-vanishing components read as

$$\begin{aligned}
J'_4 \;&=\; \frac{\mathrm{i}}{2\sqrt{2}Y^2}(1+Y^2)(a\,\mathrm{d}X_2 + b\,{*}\mathrm{d}X_2) - \sqrt{2}\mathrm{i}\,\mathrm{d}X_2,\\
J'_5 \;&=\; \frac{1}{2\sqrt{2}Y^2}(1+Y^2)(a\,\mathrm{d}X_1 + b\,{*}\mathrm{d}X_1) - \sqrt{2}\,\mathrm{d}X_1,\\
J'_6 \;&=\; \frac{\mathrm{i}}{2\sqrt{2}Y^2}(1+Y^2)(a\,\mathrm{d}X_3 + b\,{*}\mathrm{d}X_3) - \sqrt{2}\mathrm{i}\,\mathrm{d}X_3,\\
J'_{10} \;&=\; \frac{\mathrm{i}}{2\sqrt{2}Y^2}(1+Y^2)(a\,\mathrm{d}X_4 + b\,{*}\mathrm{d}X_4) + \sqrt{2}\mathrm{i}\,\mathrm{d}X_4,\\
J'_{11} \;&=\; \frac{1}{2\sqrt{2}Y^2}(1-Y^2)(a\,\mathrm{d}X_2 + b\,{*}\mathrm{d}X_2) + \sqrt{2}\,\mathrm{d}X_2,\\
J'_{12} \;&=\; \frac{\mathrm{i}}{2\sqrt{2}Y^2}(1-Y^2)(a\,\mathrm{d}X_1 + b\,{*}\mathrm{d}X_1) - \sqrt{2}\mathrm{i}\,\mathrm{d}X_1,\\
J'_{13} \;&=\; \frac{1}{2\sqrt{2}Y^2}(1-Y^2)(a\,\mathrm{d}X_3 + b\,{*}\mathrm{d}X_3) + \sqrt{2}\,\mathrm{d}X_3,\\
J'_{14} \;&=\; \frac{\mathrm{i}}{\sqrt{2}Y}(a\,\mathrm{d}Y + b\,{*}\mathrm{d}Y),\\
J'_{15} \;&=\; \frac{1}{2\sqrt{2}Y^2}(1-Y^2)(a\,\mathrm{d}X_4 + b\,{*}\mathrm{d}X_4) + \sqrt{2}\,\mathrm{d}X_4.
\end{aligned} \tag{3.43}$$

Now the T-duality can easily be implemented by using[17]

$$\mathrm{d}X_a \;=\; Y^2{*}\mathrm{d}\tilde{X}_a, \tag{3.44}$$

which yields the T-dual metric

$$\tilde{g} \;=\; Y^2\mathrm{d}\tilde{X}_a \otimes \mathrm{d}\tilde{X}_a + \frac{1}{Y^2}\mathrm{d}Y \otimes \mathrm{d}Y. \tag{3.45}$$

[16] More concretely, let $Z_{1,\ldots,6}$ be coordinates on $\mathbb{R}^{5,1}$. Euclidean AdS_5 can be viewed as the hyper-surface $Z_1^2 + \cdots + Z_5^2 - Z_6^2 = -1$ in $\mathbb{R}^{5,1}$. The relation between embedding and Poincaré coordinates, suitable for our present purposes, is then $Z_2 = X_2/Y$, $Z_3 = X_1/Y$, $Z_4 = X_3/Y$, $Z_5 = X_4/Y$, $Z_6 + Z_1 = 1/Y$ and $Z_6 - Z_1 = [Y^2 + X_{(0)} \cdot X_{(0)}]/Y^2$. To get real currents in Minkowski AdS_5, one sends X_1 to $\mathrm{i}X_1$ and finally does a reparametrization of the embedding coordinates according to $Z_3 \leftrightarrow Z_4$. Alternatively, one may directly start with the currents (3.36) in Minkowski signature, go through the procedure described in the main text and verify explicitly that the g_m-matrices are real.

[17] Notice that we work with a Minkowski world-sheet. If one instead considers a Euclidean world-sheet, one has $\mathrm{d}X_a = \mathrm{i}Y^2{*}\mathrm{d}\tilde{X}_a$. Then the T-dual currents will be real, i.e. $\mathfrak{so}(5,1)$-valued, if we choose the relation between the Poincaré and embedding coordinates as in footnote 16.

This leads to the following expressions for the T-dual currents in (3.30): $\tilde{J}(Y,\tilde{X}_a) = J(Y, X_a(\tilde{X}_b)) = \tilde{J}_A T^A$ with

$$
\begin{aligned}
\tilde{J}_4 &= \frac{\mathrm{i}}{2\sqrt{2}}(1+Y^2)(a\,{*\mathrm{d}}\tilde{X}_2 + b\,\mathrm{d}\tilde{X}_2) - \sqrt{2}\mathrm{i}Y^2{*\mathrm{d}}\tilde{X}_2,\\
\tilde{J}_5 &= \frac{1}{2\sqrt{2}}(1+Y^2)(a\,{*\mathrm{d}}\tilde{X}_1 + b\,\mathrm{d}\tilde{X}_1) - \sqrt{2}Y^2{*\mathrm{d}}\tilde{X}_1,\\
\tilde{J}_6 &= \frac{\mathrm{i}}{2\sqrt{2}}(1+Y^2)(a\,{*\mathrm{d}}\tilde{X}_3 + b\,\mathrm{d}\tilde{X}_3) - \sqrt{2}\mathrm{i}Y^2{*\mathrm{d}}\tilde{X}_3,\\
\tilde{J}_{10} &= \frac{\mathrm{i}}{2\sqrt{2}}(1+Y^2)(a\,{*\mathrm{d}}\tilde{X}_4 + b\,\mathrm{d}\tilde{X}_4) + \sqrt{2}\mathrm{i}Y^2{*\mathrm{d}}\tilde{X}_4,\\
\tilde{J}_{11} &= \frac{1}{2\sqrt{2}}(1-Y^2)(a\,{*\mathrm{d}}\tilde{X}_2 + b\,\mathrm{d}\tilde{X}_2) + \sqrt{2}Y^2{*\mathrm{d}}\tilde{X}_2,\\
\tilde{J}_{12} &= \frac{\mathrm{i}}{2\sqrt{2}}(1-Y^2)(a\,{*\mathrm{d}}\tilde{X}_1 + b\,\mathrm{d}\tilde{X}_1) - \sqrt{2}\mathrm{i}Y^2{*\mathrm{d}}\tilde{X}_1,\\
\tilde{J}_{13} &= \frac{1}{2\sqrt{2}}(1-Y^2)(a\,{*\mathrm{d}}\tilde{X}_3 + b\,\mathrm{d}\tilde{X}_3) + \sqrt{2}Y^2{*\mathrm{d}}\tilde{X}_3,\\
\tilde{J}_{14} &= \frac{\mathrm{i}}{\sqrt{2}Y}(a\,\mathrm{d}Y + b\,{*\mathrm{d}}Y),\\
\tilde{J}_{15} &= \frac{1}{2\sqrt{2}}(1-Y^2)(a\,{*\mathrm{d}}\tilde{X}_4 + b\,\mathrm{d}\tilde{X}_4) + \sqrt{2}Y^2{*\mathrm{d}}\tilde{X}_4.
\end{aligned}
\tag{3.46}
$$

As in the AdS_2 case, these currents are flat since (3.44) holds on-shell.

Remarks. The above derivation can be extended to sigma models on any AdS_p space; one simply adds the additional currents to the above set and performs successive gauge transformations.

It should be noted that one may gauge away say only the X_1 coordinate dependence to perform the T-duality only along the X_1-direction. In this case, the isometry group of the T-dual model is only a subgroup of $\mathrm{SO}(5,1)$. Nevertheless, the T-dual currents lead to the full $\mathrm{SO}(5,1)$ symmetry,[18] which we may denote, as in the AdS_2 case, by $\widetilde{\mathrm{SO}(5,1)}$. Hence, even though the isometry group of the T-dual space is smaller, the T-dual model always has a "hidden" $\widetilde{\mathrm{SO}(5,1)}$ symmetry. This is in the same spirit as in the S^2 example discussed in section 3.1 and shows again that, under the T-duality, local (Noether) charges of the original model are mapped to non-local charges of the T-dual model and vice versa. However, unlike the S^2 example, after T-duality along all X_a-directions, one recovers the same maximal symmetry group, now being generated by the Noether charges of the T-dual model.

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[18] More precisely it is $SO_{\mathbb{C}}(5,1)$, since here we are using the Euclidean signature.

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